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Stability Analysis of Non-linear Dynamical Homogeneous Systems Based on Lyapunov Function Constructed of Linear Combination of Basic Functions

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Abstract: This paper introduces a novel approach to the stability analysis of nonlinear systems using the Lyapunov function candidates. This approach focuses on the negative definiteness of the derivative of the Lyapunov function candidate, instead of positive-definiteness of the function itself. Determining such a Lyapunov function candidate, and with a sign test on Lyapunov function, the stability of zero equilibrium state (ZES) of the systems is assessed. In this new approach, a Lyapunov function candidate is constructed as a linear combination of some basic functions. The coefficients of this linear combination have to be determined such that the derivative of the resulting Lyapunov candidate becomes negative definite. Also, an algebraic approach to form such linear combination for zero order homogeneous polynomial systems is provided which generalizes the Lyapunov equation in the stability analysis of the linear systems.
Keywords: Lyapunov function, stability analysis, zero equilibrium state (ZES), generalized Lyapunov equation, non-linear dynamical homogeneous systems.

INTRODUCTION

Industrial systems are often described as dynamical systems. Stability is a property for these systems that is important in several aspects: at first the safety of systems depends on keeping some critical variables within the acceptable range, this is known as stability of dynamical systems (Khalil, 2002). Secondly, the desired performances of these systems is usually expressed as following of main parameters from desired patterns. The quantity of this is expressed as stability conditions (Slotine and Li, 1991). The stability analysis for dynamical systems is the most important subject in this systems. So it is very important to express efficient methods determining stability or instability of these systems.

In linear time invariant (LTI) systems there are tools expressing necessary and sufficient conditions for the stability. Eigenvalues and Lyapunov equation are two such tools. They are useful because at first they are used in finite algorithms. Secondly stability or instability is achieved for a system and this is due to necessary and sufficient conditions.

Unfortunately, in nonlinear or time variant systems there are not any tools with above benefit. Tools for stability analysis of these systems have infinite algorithm, means they are used with trial and error that is not known how many times must be repeated. Secondly, the tools express only sufficient conditions for stability or instability of systems. So we can't remark about stability or instability if we can't find these sufficient conditions. It means that is the starting point again.

A main tool for stability analysis of nonlinear systems is the Lyapunov theorem that expresses a sufficient condition for equilibrium point stability as existence of a Lyapunov function $V(x)$. The Lyapunov function $V(x)$ is a scalar positive definite function, whose time derivative along the trajectories of a system is negative definite. Usually this tool is used in a trial and error process, it means that you must choose a positive definite function at first, then calculate its time derivative along the system trajectories. If the time derivative is negative definite, then stability is proved for the equilibrium point, otherwise we can't comment on stability or instability. So we should begin with another positive definite function and repeat the process. Another way to find a Lyapunov function is the variable gradient method. In this method first choose a general form with unknown coefficients for gradient function $\nabla V(x)$ and exert the symmetry condition to be a well-defined function. The choice of gradient function is effective in the success of method, but in complex systems there isn't a specific way to choose it. Then computing the time derivative function from gradient, its unknown coefficients must be calculated such that the derivative function be negative definite. However, in complex systems there is not a specific method to make it negate definite. After computing unknown coefficients, $V(x)$ is calculated by integration from its gradient. In this step if $V(x)$ is positive definite so it is a Lyapunov function and proves the stability.

Assume the zero equilibrium point for a nonlinear system is unstable, so the above effort is not successful to construct a Lyapunov function. The instability theorems may help us in this situation. They express sufficient conditions for equilibrium point to be unstable. One of such theorems expresses that if $\dot{V}(x)$ is positive definite but $V(x)$ be indefinite or positive definite then the zero equilibrium point is unstable.

As it was shown above, efforts for proving stability or instability are done separately, because the respective theorems have been expressed in sufficient conditions. In this article, the common point in stability and instability theorems that is the sign definiteness of $\dot{V}(x)$ is considered and a stability/instability theorem is expressed. Using this new theorem all efforts are done to make a negative definite $\dot{V}(x)$. If we find a negative definite $\dot{V}(x)$, then stability or instability of zero equilibrium point is determined with a sign test for $V(x)$.

A tool for using the new theorem is the variable gradient method (Slotine and Li, 1991), introduced above. Since this method makes a negative definite $\dot{V}(x)$ at first, then finds $V(x)$ by integration. So at last we can check the sign of $V(x)$ and determine stability or instability. But this method has some drawbacks: at first, the symmetry condition for gradient is complex. In second, it is hard to make $\dot{V}(x)$ negative definite and this is usually done algebraically. In third, $V(x)$ is calculated by integration.

In this article a new method is presented to use the new theorem. It does not have drawbacks of the variable gradient method. This method assumes some basic functions then makes the Lyapunov function by a linear combination of basic functions with unknown coefficients as $V(x)=\sum_{i=1}^{m} a_i v_i(x)$. So the derivative of Lyapunov function $\dot{V}(x)=\sum_{i=1}^{m} a_i \dot{v}_i(x)$ is calculated easily. Since coefficients are unknown, we can find them by solving some linear matrix inequalities for $\dot{V}(x)$ which is negative definite. The solution of inequalities can be algebraic or numeric. After finding coefficients, $V(x)$ can be found without integration eventually.

This method runs in a finite process and its success relates to richness of choosing basic functions. For example, in nonlinear polynomial systems choosing basic functions as monomials up to a certain degree can be helpful. The new method can be implemented easier in the special class of homogeneous nonlinear systems (Rosier, 1992). In the particular class of homogeneous systems with zero degree, the new method is a perfect method as same as Lyapunov method in LTI systems. It means that the new method runs in a finite process and determines stability or instability for this special class.

In this paper, we consider the stability analysis of the following time-invariant system with zero equilibrium state (ZES).

$$\dot{x}=\mathrm{f}(x) \quad , \quad x \in \mathbb{R}^n \tag{1}$$

Some methods of constructing Lyapunov functions that are available in previous research literature, are as follows:

The variable gradient method (VGM) is a known method for constructing Lyapunov functions. This method begins by selecting a general form for gradient function ∇V, which must satisfy the condition curl of $\partial \nabla v_i / \partial x_j = \partial \nabla v_j / \partial x_i$ for $i,j=1,\ldots,n$. Then, the exact value of ∇V is found so that a given function $\dot{V}(x)=[\nabla V]^T f(x)$ is negative definite. Finally, $V(x)$ is calculated by integrating of ∇V (according to Reference (Slotine and Li, 1991)).

Another method of constructing the Lyapunov functions, uses Sum of Squares (SOS) techniques (Papachristodoulou and Prajna, 2005). The SOS techniques are numerical calculation methods and usually they are used to construct polynomial Lyapunov functions. In these techniques Lyapunov function $V(x)$ (or its negative derivative $-\dot{V}(x)$) is written in the form of the sum of squares $V(x)=\sum_{i=1}^{m} h_i^2(x)$. In research (Papachristodoulou and Prajna, 2002), the construction of polynomial Lyapunov functions using the SOS techniques has been developed to Lyapunov functions include a term for non-polynomial and a few examples have been solved. Also in articles (Fisher and Bhattacharya, 2007) and (Fisher and Bhattacharya, 2009), the construction algorithms of Lyapunov functions using the SOS techniques to solve asymptotically stability problems are provided and some examples are given.

Recently, we have presented a method for constructing a Lyapunov function of a linear combination of a positive definite function and its higher order derivatives in article (Meigoli and Nikravesh, 2012). This method was complicated because higher order derivatives of $V(x)$ or their approximate should have been calculated. In this paper, this idea is extended and a Lyapunov function is formed as a linear combination of basic functions such as $V(x)=\sum_{i=1}^{m} a_i v_i(x)$. In this case, the basic functions can be monomial functions and $V(x)$ can be a polynomial function with unknown coefficients of a_i. It is a new approach to find and construct a Lyapunov function candidate for stability analysis of non-linear dynamical systems.

Theorems And Methods
Mathematical Preliminaries

In this paper, a special class of time-invariant nonlinear systems, namely homogeneous systems, is analyzed for stability. In the following comes the definitions and basic theorems on these systems (Kawski, 1991).

For a sequence of weights $r=(r_1,\ldots,r_n)$ for $r_i \in \mathbb{R}$ and a non-negative variable ($\lambda \geq 0$) a dilation is defined as following linear transformation.

$$\begin{cases} \Delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n \\ \Delta_\lambda^r(x) \triangleq (\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n) \end{cases} \tag{2}$$

A scalar function *V(x)* is homogeneous of degree *p* with respect to dilation Δ_λ^r and it can be written as $v \in H_p$ if $v(\Delta_\lambda^r x)=\lambda^p v(x)$. Similarly, a system $\dot{x}=\mathrm{f}(x)$ is homogeneous of degree *k* with respect to dilation Δ_λ^r and it can be mentioned as $\mathrm{f} \in n_k$ if $\mathrm{f}(\Delta_\lambda^r x)=\lambda^k \Delta_\lambda^r \mathrm{f}(x)$, or in other words, $\mathrm{f}_i \in H_{r_i+k}$ for $i=1,\ldots,n$.

Lemma 1. (Kawski, 1991) If $v \in H_p$ and $\mathrm{f} \in n_k$ then derivative of *V(x)* along the solutions of system $\dot{x}=\mathrm{f}(x)$ is a homogeneous function of degree $p+k$ that is $\dot{v} \in H_{p+k}$.

Lemma 2. (Rosier, 1992) Suppose the system introduced in equation (1) is continuous and homogeneous with respect to the dilation of the equation (2). If the system's ZES is asymptotically stable, then there is a homogeneous Lyapunov function over the same dilation for the stability analysis of the system.

Example 1. It is clear that the function $v(x)=a_1x_1x_2^2+a_2x_1^4x_2+a_3x_1^7$ with sequence of weights as $(r_1,r_2)=(1,3)$ is homogeneous of degree 7 ($v \in H_7$) and the system presented in equation (3) is homogeneous of degree 2 ($\mathrm{f} \in n_2$). The proofs are presented in equations (4) and (5);

$$\begin{cases} \dot{x}_1 = b_{11}x_1^3 + b_{12}x_2 \\ \dot{x}_2 = b_{21}x_1^5 + b_{22}x_1^2x_2 \end{cases} \tag{3}$$

$$\begin{aligned} V(\Delta_y^r x) &= V(\lambda x_1, \lambda^3 x_2) \\ &= a_1\lambda x_1(\lambda^3 x_2)^2 + a_2(\lambda x_1)^4(\lambda^3 x_2) + a_3(\lambda x_1)^7 \\ &= \lambda^7(a_1x_1x_2^2 + a_2x_1^4x_2 + a_3x_1^7) = \lambda^7 V(x) \end{aligned} \tag{4}$$

$$\begin{aligned} f(\Delta_\lambda^r x) &= \begin{bmatrix} -2(\lambda x_1)^3 + \lambda^3 x_2 \\ 3(\lambda x_1)^5 - (\lambda x_1)^2(\lambda^3 x_2) \end{bmatrix} \\ &= \lambda^2 \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{bmatrix} \begin{bmatrix} -2x_1^3 + x_2 \\ 3x_1^5 - x_1^2 x_2 \end{bmatrix} = \lambda^2 \Delta_\lambda^r f(x) \end{aligned} \tag{5}$$

Derivative of *V* (*x*) along with system solutions of (3) can be calculated as below:

$$\begin{aligned} \dot{V}(x) &= a_1(-2x_1^3x_2^2 + x_2^3 + 6x_1^6x_2 - 2x_1^3x_2^2) \\ &+ \mathrm{a}_2(-8x_1^6x_2 + 4x_1^3x_2^2 + 3x_1^9 - x_1^6x_2) + a_3(-14x_1^9 + 7x_1^6x_2) \end{aligned} \tag{6}$$

Also *$\dot{V}$(x)* is homogeneous of degree 9 ($\dot{v} \in H_9$).

The homogeneous systems have a special type of radial symmetry which leads to the equivalence of their local properties around the origin and their global properties. Due to their common and similar properties with linear systems, such systems can be an appropriate platform to practice the suitable approaches of linear systems for nonlinear systems as well.

Example 2. With weights (*r* = (1, 2, 3)) the following system is homogeneous of degree zero.

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = 2x_1^2 - 3x_2 \\ \dot{x}_3 = 2x_1^3 - 4x_1x_2 - 2x_3 \end{cases} \tag{7}$$

According to Lemma 2, some Lyapunov function of some degree *p* is sufficient for stability analysis of such system. But how is selected the suitable *p*? It is obvious that *p* must be a common factor of the weights r_i s. For example, a general homogeneous Lyapunov function candidate of degree 6 for this system can be introduced by the following equation:

$$V(x) = a_1 x_1^6 + a_2 x_1^4 x_2 + a_3 x_1^3 x_3 + a_4 x_1^2 x_2^2 + a_5 x_1 x_2 x_3 + a_6 x_2^3 + a_7 x_3^2 \quad (8)$$

Obviously, this Lyapunov function candidate cannot be positive definite, because substituting $x_1 = x_3 = 0$ in the above equation leads to $V(x) = a_6 x_2^3$ which alters its sign. Hence, the lowest degree possible for a Lyapunov function for this system is *p = 12*.

Theorems based on the sign of the $\dot{V}(x)$ function

The most famous theorem in the stability analysis of nonlinear systems is Lyapunov theorem (regarded as Lyapunov direct method) which provides sufficient conditions for stability.

Theorem 1. (Slotine and Li, 1991) Lyapunov direct method: If there is a continuous positive definite function *V(x) i.e.*:

$$\begin{cases} V(x) > 0 & \forall x \neq 0 \\ V(0) = 0 \end{cases} \quad (9)$$

Whose derivative along the solutions of equation (1) is negative definite:

$$\dot{V}(x) = [\nabla V]^T f(x) < 0, \quad \forall x \neq 0 \quad (10)$$

Then the ZES of system equation (1) is asymptotically stable.

Finding a Lyapunov function is always a major challenge for control engineers. If after selecting a certain positive function *V(x)*, its derivative $\dot{V}(x)$ does not become negative definite or even negative semi-definite, then it is not helping with the stability of system's ZES. The reason lies where the Lyapunov's direct method proposes sufficient but not necessary conditions for the stability of the equilibrium state.

Note that, for autonomous systems, one might think that the Lyapunov's linearization method is sufficient for the study of unstability. However, in some cases, unstability theorems based on the direct method may be advantageous. Indeed, if the linearization method fails (i.e., if the linearized system is marginally stable), the following theorem may be used to determine the unstability of the non-linear system.

Theorem 2. (Slotine and Li, 1991) Unstability theorem: If there is a continuous differentiable function *V(x)* for equation (1) in a certain neighborhood Ω around the origin, such that following conditions are met:

V (0) = 0

V(x) contains positive values in the area Ω_1 of Ω as presented in **Figure 1** and origin is located on boundary of Ω_1.

$\dot{V}(x)$ is positive definite in Ω.

Then ZES of the system is unstable.

In the following a theorem will be expressed that presents a distinct approach to the use of Theorem 1.

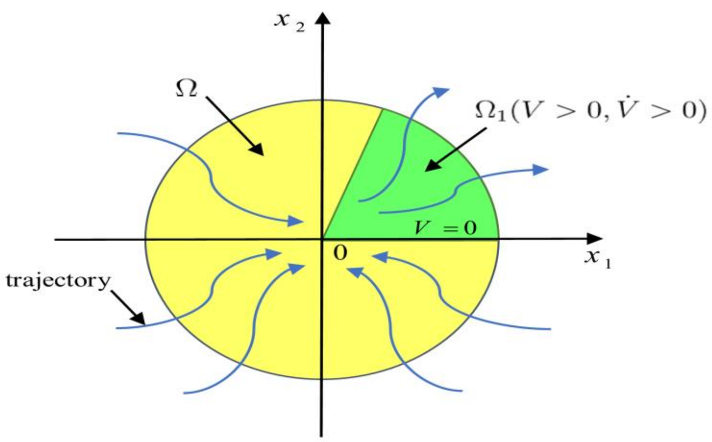

Figure 1. Interest areas in the unstability theorem (Theorem 2)

Theorem 3. (Inverse Lyapunov theorem): Let ZES of equation (1) is asymptotically stable. If there is a continuous function *V(x)* such that *V (0) = 0* and $\dot{V}(x)$ is negative definite then *V(x)* is positive definite.
Proof. For a dynamic autonomous and continuous system (1) that has a stable equilibrium at the origin the derivative of an arbitrary function *V(x)* is written as following:

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) \tag{11}$$

Now with the integration of both sides, we have:

$$\int_0^{+\infty} \dot{V}(x) dt = \int_0^{+\infty} \frac{d}{dt} V(x(t)) dt \tag{12}$$

Which gives:

$$\int_0^{+\infty} \dot{V}(x) dt = \lim_{t \to +\infty} V(x(t)) - V(x(0)) \tag{13}$$

Assuming that system is asymptotically stable $(\lim_{t\to\infty} x(t) = 0)$ and $\dot{V}(x(t)) < 0$, we conclude the following equation:

$$V(x(0)) = \int_0^{+\infty} -\dot{V}(x(t)) dt \tag{14}$$

Which illustrates the positive definiteness of the function $V(x(0))$ at any starting point outside the origin.
Combining the previous theorems leads to the following theorem:

Theorem 4. (Main Theorem): If there is a continuous function *V(x)* such that *V* (0) = *0* and $\dot{V}(x)$ is negative definite, then exactly one and only one of the three following conditions holds:
V(x) is a locally positive definition function and hence the ZES of system (1) is asymptotically stable.
V(x) is a locally negative definite and hence the ZES of system (1) is unstable.
The sign of *V(x)* changes in every neighbourhood of the origin and hence ZES of system (1) is unstable.
Proof. Obviously *V(x)* must be one of positive definite, negative definite or indefinite, because any conditions other than the these three conditions cannot happen, (i.e., *V(x)* can not be negative or positive semi-definite). As a contradiction method let *V(x)* be positive semi definite and $V(x_0) = 0$ at a point out of the origin. Since $\dot{V}(x_0) < 0$, then the sign of *V(x)* changes when the trajectory crosses this point. This result disagrees the semi-definiteness of the function. Now we consider all the three conditions:
case 1. In this case the Theorem 1 establishes the asymptotic stability.
case 2. Theorem 2 can be utilized for proof of this case. Multiplying *(−1)* into *V(x)* and $\dot{V}(x)$ to alter their signs makes them positive definite in a neighbourhood Ω_1 of the origin and provides conditions of Theorem 2, so the ZES of system becomes unstable.
case 3. For this case, consider a region near the origin, namely Ω, in which we alter the sign of *V(x)*, but $\dot{V}(x) < 0$. Let Ω_1 be points where *V(x) < 0*. Again multiplying *V(x)* and $\dot{V}(x)$ by *(−1)* such that the conditions of Theorem 2 hold and the unstability for ZES of system can be proved.

Initiate from the derivative of Lyapunov

In the conventional approach to stability analysis of system (1), at first, a positive definite Lyapunov function candidate *V(x)* is selected followed by calculation of $\dot{V}(x)$ and if it would not become negative definite, it does not determine stability or unstability of the system. Following Theorem 4, a novel and new approach to stability analysis is introduced such that our efforts are initiated to make the $\dot{V}(x)$ function negative definite. Afterwards, by determining the sign of the *V(x)* function, the stability or unstability of ZES follows. Approaches to find a negative definite $\dot{V}(x)$ and utilitzing the Theorem 4 are briefly discussed in this section.

Direct selection of $\dot{V}(x)$ function

Application of Theorem 3 is the simplest method to construct a Lyapunov function, i.e. selecting a $\dot{V}(x) < 0$ and then using equation (14). However, using this relation requires a pre-knowledge about the analytical solution of the system, which is easy to obtain only in linear systems.
Example 3. Consider the following linear system:

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_2 \end{cases} \tag{15}$$

We know the system solution for $t \geq 0$ is as following:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{16}$$

Choosing an arbitrary $\dot{V}(x)$ which is negative definite:

$$\dot{V}(x(t)) = -x_1^2(t) - x_2^2(t) \tag{17}$$

and using equation (14) gives:

$$V(x(0)) = \int_0^{+\infty} -\dot{V}(x(t))dt == \int_0^{+\infty} \left[x_1^2(t) + x_2^2(t) \right] dt$$

$$= \int_0^{+\infty} \left\{ \left[e^{-t} x_1(0) + te^{-t} x_2(0) \right]^2 + \left[e^{-t} x_2(0) \right]^2 \right\} dt \tag{18}$$

Calculating this integral, yields:

$$V(x(0)) = \frac{1}{2} x_1^2(0) + \frac{1}{2} x_1(0) x_2(0) + \frac{3}{4} x_2^2(0) \tag{19}$$

Therefore, according to arbitrariness of $x(0)$, the following positive definite function is achieved:

$$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_1 x_2 + \frac{3}{4} x_2^2 \tag{20}$$

Likewise, for systems' stability analysis we can always start with a negative definite $\dot{V}(x)$ rather than a positive definite $V(x)$.

Variable gradient method

The variable gradient method is a common approach to find a Lyapunov function. This method constructs a Lyapunov function starting with a certain form for its gradient with unknown coefficients followed by identifying Lyapunov function by integrating its gradient (Slotine and Li, 1991). The relationship between $V(x)$ and its gradient is as follows:

$$V(x) = \int_0^x \nabla V \, dx \tag{21}$$

Where $\nabla V = \left[\partial V / \partial x_1, ..., \partial V / \partial x_n \right]^T$. Hence, in order to recover a unique scalar function V from the gradient ∇V, the gradient function has to satisfy the following curl conditions:

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}, \ (i, j = 1, 2, ..., n) \tag{22}$$

The principle of the variable gradient method is to assume a specific form for the ∇V, instead of assuming a specific form for the Lyapunov function V itself. Usually the form of this gradient function is as follows:

$$\nabla V_i = \sum_{j=1}^{n} a_{ij}(x) x_j \tag{23}$$

Where a_{ij} s are generally non constant coefficients to be determined. The Lyapunov function can be found by this method through the following steps:

Assume that ∇V is given by equation (23) (or any other suitable form).

Solve for the a_{ij} coefficients such that the curl equations in equation (22) be satisfied and $\dot{V}(x) = [\nabla V]^T f(x)$ becomes globally (or even locally) negative definite .

Calculate $V(x)$ from equation (21).

Check whether $V(x)$ is positive definite or not?

Example 4. Calculate a Lyapunov function by using the above mentioned method for the following homogeneous system.

$$\begin{cases} \dot{x}_1 = -x_1^3 + 2x_2^3 \\ \dot{x}_2 = x_1^3 - 3x_2^3 \end{cases} \tag{24}$$

First, we assume the gradient function as follows:

$$\nabla V = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \delta(x)x_1 + \gamma(x)x_2 \end{bmatrix} \tag{25}$$

Then $\dot{V}(x)$ can be computed as:

$$\begin{aligned} \dot{V}(x) &= \nabla V^T f(x) \\ &= \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \delta(x)x_1 + \gamma(x)x_2 \end{bmatrix}^T \begin{bmatrix} -x_1^3 + 2x_2^3 \\ x_1^3 - 3x_2^3 \end{bmatrix} \\ &= (-\alpha(x) + \delta(x))x_1^4 + (2\beta(x) - 3\gamma(x))x_2^4 \\ &+(-\beta(x) + \gamma(x))x_1^3 x_2 + (2\alpha(x) - 3\delta(x))\, x_1 x_2^3 < 0 \end{aligned} \tag{26}$$

Also, the symmetry (Curl) condition must be met which requires:

$$x_1 \frac{\partial\alpha(x)}{x_2} + \beta(x) + x_2 \frac{\partial\beta(x)}{x_2} = x_1 \frac{\partial\delta(x)}{x_1} + \delta(x) + x_2 \frac{\partial\gamma(x)}{x_1} \tag{27}$$

Since the function in equation (26) must be negative definite, we can set $\gamma - \beta = 0,\ 2\alpha - 3\delta = 0,\ \alpha - \delta > 0,\ 3\gamma - 2\beta > 0$. Considering the condition in equation (27), the parameters $\beta = \delta = \frac{2}{3}\alpha = c > 0$ are chosen as positive constants.

$$\nabla V = \begin{bmatrix} \frac{3}{2}cx_1 + cx_2 \\ cx_1 + cx_2 \end{bmatrix} \tag{28}$$

Now applying the equation (21) results in *V(x)*.

$$\begin{aligned} V(x) &= \int_0^{x_1} (\frac{3}{2}cy_1 + c \times 0)dy_1 + \int_0^{x_2} (cx_1 + cy_2)dy_2 \\ &= c\frac{3}{4}x_1^2 + cx_1x_2 + c\frac{1}{2}x_2^2 \end{aligned} \tag{29}$$

Some of the disadvantages of the variable gradient method that are illustrated in the above example, are:
It requires integration to deliver *V(x)*.
In above example, the coefficients of the gradient function were assumed to be constant which it is not the case in general form and this complicates the identification of a negative definite $\dot{V}(x)$.
The symmetry condition (27) in this method is complicated.
The calculation process is only possible analytically.
This method is more commonly used in low rank systems and there is a possibility that a Lyapunov function is not achieved.

Solving the Lyapunov equation

Solving the Lyapunov equation method is more common in linear systems. For the following linear system:

$$\dot{x}(t) = Ax(t) \tag{30}$$

A square Lyapunov function as presented below is selected:

$$V(x) = x^T Px \tag{31}$$

This method, called second Lyapunov method, is applied in linear systems or nonlinear systems after linearization. In this method a Lyapunov equation in the following form must be solved for *P* using a positive definite matrix *Q* :

$$A^T P + PA = -Q \tag{32}$$

Usually *Q* is selected as unity matrix. Then the equation is solved for *P* and asymptotic stability is proved for the ZES of system if *P* is a positive definite matrix.

Theorem 5. (Khalil, 2002) If ZES for the linear system (30) is asymptotically stable, then Lyapunov equation in equation (32) has a unique solution and P becomes positive definite. Hence *V(x)* is a suitable Lyapunov function for proving asymptotic stability.

Lyapunov equation in equation (32) is achieved by differentiation from *V(x)* in equation (31) as below:

$$\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x} = (Ax)^T Px + x^T P(Ax)$$
$$= x^T (A^T P + PA)x \triangleq -x^T Qx \qquad (33)$$

As the result of equation (33), the idea of choosing a positive definite Q means starting from a negative definite $\dot{V}(x)$. Certainly, one of the best options for Q is the identity matrix, and calculating P from Lyapunov equation means calculation of $V(x)$ in an algebraic approach. Also, choosing an unknown P matrix means choosing a square Lyapunov function with unknown coefficients. Calculation of these coefficients is performed by identifying the $\dot{V}(x)$ function. So, this main idea is generalized in this paper for nonlinear systems by introducing a general Lyapunov function with unknown coefficients.

The linear combination of basic functions

To propose the new approach of this paper, it is assumed that a Lyapunov function for the stability analysis of ZES of system (1) exists in the form of a linear combination of some basic functions as follows:

$$V(x) = \sum_{i=1}^{m} a_i v_i(x) \qquad (34)$$

These basic functions must be linearly independent for the coefficients of the above mentioned linear combination to be unique. Moreover, all stability or instability theorems require the $V(0) = 0$ condition and therefore all basic functions are assumed to be zero at the origin.

$$v_i(0) = 0 \quad , \quad \forall i \qquad (35)$$

The basic functions $v_i(x)$ are assumed to be predefined and known and their derivatives are easily calculated as $\dot{v}_i(x) = \left[\nabla v_i\right]^T f(x)$. Our proposed method suggests the calculation of the unknown a_i coefficients such that the following function becomes negative definite (at least locally).

$$\dot{V}(x) = \sum_{i=1}^{m} a_i \dot{v}_i(x) \qquad (36)$$

After calculating the coefficients, there is no need for integration to calculate $V(x)$ and stability or instability of the system is determined from Theorem 4. Although the basic functions are linearly independent, but their derivatives might be linearly dependent. In this case since the a_i coefficients are obtained from equation (36), these coefficients are not unique. The next theorem studies this situation.

Theorem 6. If the basic functions $v_i(x)$ in equation (34) are linearly independent and also the condition in equation (35) holds but the derivatives of the basic functions are linearly dependent for a special system, then ZES of that system cannot be asymptotically stable.

Proof. Suppose the conditions in Theorem 6. So for some $a_i \neq 0$:

$$\dot{V}(x) = \sum_{i=1}^{m} a_i \dot{v}_i(x) = 0 \quad \Rightarrow \quad V(x) = const \qquad (37)$$

It means that the Lyapunov function candidate $V(x)$ is constant during any solutions of equation (1). Basic functions are independent and thus $V(x)$ is not identically zero in the neighborhood of origin, whereas $V(0) = 0$. Therefore, a solution very close to the origin which begins from a $V(x_0) \neq 0$, cannot converge to the origin, where $V(0)$ vanishes. So, ZES for equation (1) cannot be asymptotically stable, but it may be stable in the sense of Lyapunov or even unstable.

The Proposed Method And Examples

Using the proposed method in class of homogeneous systems

In this section, we assume that system (1) is polynomial homogeneous of degree k with respect to the given dilation in equation (2) ($f \in n_k$). It is expected from Lemma 2 that a homogeneous polynomial function $V(x)$ of degree p and with respect to the same dilation is sufficient for ZES stability analysis. To apply the suggested method of this paper, the basic functions are chosen as a collection of all monomials of homogeneous functions of degree p ($v_i \in H_p$). Therefore, the collection of this basic functions makes a basis for the H_p vector space (collection of all homogeneous polynomials of degree p). We choose these basic functions as monomial expressions $v_i(x) = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ such that:

$$p_1 r_1 + p_2 r_2 + \cdots + p_n r_n = p \quad , \quad p_j \geq 0 \qquad (38)$$

Hence, the number of basic functions (m) is equal to the number of non-negative solutions of equation (38). For the set of homogeneous polynomials of degree p can include positive definite or negative definite functions, H_p should contain monomial expressions such as $x_j^{2k_j}$ for $j = 1, ..., n$. Substituting this for equation

(38) we have $p = 2r_j k_j$ meaning that *p* should be divisible by $2r_j$. So, we can choose *p* twice as the lowest common multiple of all weights r_j.

For example for *r* = (1, 2, 3), we choose *p* = 12 and the set of basic functions for these values are to be selected as follows:

$$v_i(x) \in \left\{ x_1^{12}, x_1^{10}x_2, x_1^9x_3, x_1^8x_2^2, x_1^7x_2x_3, x_1^6x_2^3, x_1^6x_3^2 \right.$$
$$x_1^5x_2^2x_3, x_1^4x_2^4, x_1^4x_2x_3^2, x_1^3x_2^3x_3, x_1^3x_3^3, x_1^2x_2^5,$$
$$\left. x_1^2x_2^2x_3^2, x_1x_2^4x_3, x_1x_2x_3^3, x_2^6, x_2^3x_3^2, x_3^4 \right\} \subset H_{12} \quad (39)$$

The order in writing these monomial expressions is such that the highest priority is given to those expressions with greater power for the first variable and then priority is given to the greater power for second variable and so on. Since the functions x_1^{12}, x_2^6, x_3^4 are in the list of suggested basic functions in equation (39), they participate in making positive definite or negative definite functions.

Assume a Lyapunov function candidate as a linear combination of above basic functions. Also applying Lemma 1 ensures that the derivatives of basic functions are homogeneous of degree *p* + *k* ($\dot{v}_i(x) \in H_{p+k}$). In application of Theorem 4, it is desirable that the linear combination of derivative functions be a negative definite function.

A negative definite linear combination in equation (36) can be achieved through numerical methods or algebraic approaches. Numerical methods will be mentioned in future works and is not in the scope of the present paper. However, an effective algebraic approach for polynomial homogeneous nonlinear systems is presented in the following.

In this section we consider a special case that (1) is a homogeneous system of zero degree with respect to the given dilation in equation (2) ($f \in n_0$). So by choosing the basic functions ($v_i(x) \in H_p$) and applying the Lemma 1, we have $\dot{v}_i(x) \in H_p$. As a result, both linear combination of basic functions in equation (34) and linear combination of their derivatives in equation (36) are homogeneous polynomial functions of degree *p* and are expandable in terms of same basic functions. We suggest that the linear combination (36) be set identical to a certain negative definite function and followed by calculating the coefficients of linear combination. We use the simplest possible negative definite function which is a homogeneous polynomial of degree *p* in the following equation:

$$\dot{V}(x) = \sum_{i=1}^{m} a_i \dot{v}_i(x) \equiv -x_1^{2k_1} - x_2^{2k_2} - \ldots - x_n^{2k_n} \quad (40)$$

Where $2r_j k_j = p$ for $j = 1,\ldots,n$. Since all functions in equation (40) are linear combinations of basic functions of degree *p* and the number of basic functions are *m*. Therefore, the equation in equation (40) becomes equivalent to a system of *m* equations with *m* unknowns a_i. Solving this system of equations (*m* × *m*) results in all unknown coefficients a_i and the negative definite function $\dot{V}(x)$ is achieved.

Afterwards, the asymptotic stability or instability of the ZES of the system (1) is determined using a sign test for equation (34).

There is a question about existence and uniqueness of solutions for system of *m* equations with *m* unknowns from equation (40) that is answered in the ahead theorem. This theorem generalizes Theorem 5 to be used for homogeneous systems of degree zero.

Theorem 7. If polynomial system (1) is homogeneous of degree zero with respect to the given dilation in equation (2) and its ZES is asymptotically stable, then the system of equations in equation (40) has unique solution set of a_i*'s* and the resulted *V(x)* is positive definite and is a suitable Lyapunov function to prove the asymptotic stability of the ZES.

Proof. First we prove the uniqueness of solution of system of equations in equation (40). The *m* numbers basic functions $v_i(x)$ are linearly independent and the ZES of equation (1) is asymptotically stable. According to Theorem 6 the derivative functions must be linearly independent. Since the number of derivative functions is also *m*, these functions form a basis for the H_p vector space. Each function in H_p space can be uniquely expanded as a linear combination of derivative functions, including the right hand function in equation (40). Therefore, the a_i coefficients in equation (40) uniquely exist. The function $\dot{V}(x)$ in equation (40) is negative definite and the ZES is asymptotically stable, thus using Theorem 3 proves that *V*(*x*) is positive definite and suitable to use Theorem 1.

As a result, the method of linear combination of basic functions for stability analysis of homogeneous polynomial systems of degrees zero seems to be a perfect method. In the next section, some examples of stability analysis of the ZES of the system in equation (1) using the new method are presented.

Provide some examples

Example 5. Consider this system with unknown b_{ij} parameters:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix} \tag{41}$$

Find inequalities for b_{ij} parameters that show ZES of the system is either asymptotically stable or unstable.

Solution: This system is homogeneous of degree 2 with respect to the weights $(r_1, r_2) = (1,1)$. We apply the following basic functions for stability analysis:

$$v_1 = x_1^2, \quad v_2 = x_1 x_2, \quad v_3 = x_2^2 \tag{42}$$

Then, we form the linear combination of their derivatives:

$$\begin{aligned} \dot{V} &= a_1\dot{v}_1 + a_2\dot{v}_2 + a_3\dot{v}_3 \\ &= a_1[2x_1(b_{11}x_1^3 + b_{12}x_2^3)] + a_2[(b_{11}x_1^3 + b_{12}x_2^3)x_2 \\ &\quad + x_1(b_{21}x_1^3 + b_{22}x_2^3)] + a_3[2x_2(b_{21}x_1^3 + b_{22}x_2^3)] \\ &= x_1^4(2a_1b_{11} + a_2b_{21}) + x_1x_2^3(2a_1b_{12} + a_2b_{22}) \\ &\quad + x_1^3x_2(a_2b_{11} + 2a_3b_{21}) + x_2^4(a_2b_{12} + 2a_3b_{22}) \end{aligned} \tag{43}$$

To facilitate the sign test for $\dot{V}(x)$, we set the coefficients of $x_1x_2^3$ and $x_1^3x_2$ equal to zero:

$$\begin{cases} 2a_1b_{12} + a_2b_{22} = 0 \\ a_2b_{11} + 2a_3b_{21} = 0 \end{cases} \Rightarrow \begin{cases} 2a_1 = -b_{22}\,a_2/b_{12} \\ 2a_3 = -b_{11}\,a_2/b_{21} \end{cases} \tag{44}$$

Substituting the results of equation (44) in (43), we have:

$$\dot{\mathrm{V}}(x) = (b_{12}b_{21} - b_{11}b_{22})\left(\frac{a_2}{b_{12}}x_1^4 + \frac{a_2}{b_{21}}x_2^4\right) \tag{45}$$

It is necessary to suppose identical signs for b_{12} and b_{21} for the uniqueness of sign of $\dot{V}(x)$ in equation (45). The Lyapunov function candidate of:

$$\mathrm{V}(x) = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 \tag{46}$$

has a unique sign only when its Δ is negative:

$$\begin{aligned} \Delta &= a_2^2 - 4a_1a_3 = a_2^2 - 4a_2^2\, b_{11}b_{22}/(4b_{12}b_{21}) \\ &= a_2^2\,(b_{12}b_{21} - b_{11}b_{22})/(b_{12}b_{21}) \end{aligned} \tag{47}$$

Comparing the equations (43)-(47) three cases can be assumed for the b_{ij} parameters:

$$\begin{aligned} &case1: \begin{cases} b_{12}b_{21} - b_{11}b_{22} < 0 \\ b_{12}b_{21} > 0, \quad b_{11} < 0, \quad b_{22} < 0 \end{cases} \Rightarrow \Delta < 0,\ a_1, a_3 > 0 \\ &case2: \begin{cases} b_{12}b_{21} - b_{11}b_{22} < 0 \\ b_{12}b_{21} > 0, \quad b_{11} > 0, \quad b_{22} > 0 \end{cases} \Rightarrow \Delta < 0,\ a_1, a_3 < 0 \\ &case3: \begin{cases} b_{12}b_{21} - b_{11}b_{22} > 0 \\ b_{12}b_{21} > 0 \end{cases} \Rightarrow \Delta > 0 \end{aligned} \tag{48}$$

The three conditions in equation (48) are corresponded to three conditions in Theorem 4. In all these three cases the sign of a_2 is selected such that negative definiteness of $\dot{V}(x)$ in equation (45) be ensured. In first case *V(x)* is positive definite and ZES of equation (41) is asymptotically stable. In second case *V(x)* is negative definite and ZES for equation (41) is unstable. In third case *V(x)* is an indefinite function and ZES for equation (41) is unstable.

Example 6. For weight *r* = (1, 3), the general form of homogeneous system of degrees zero is as follows:

$$\begin{cases} \dot{x}_1 = b_1 x_1 \\ \dot{x}_2 = b_2 x_1^3 + b_3 x_2 \end{cases} \tag{49}$$

Use homogeneous basic functions of degree *p* = 6 and the following Lyapunov function for stability analysis of ZES of this system:

$$\mathrm{V}(x) = a_1 x_1^6 + a_2 x_1^3 x_2 + a_3 x_2^2 \tag{50}$$

Then the derivative of the Lyapunov function candidate is calculated and is set identical to a certain negative definite function:

$$\begin{aligned} \dot{\mathrm{V}}(x) &= a_1 6 x_1^5 \dot{x}_1 + a_2 [3x_1^2 x_2 \dot{x}_1 + x_1^3 \dot{x}_2] + a_3 2 x_2 \dot{x}_2 \\ &= a_1 6 x_1^5 b_1 x_1 + a_2 [3x_1^2 x_2 b_1 x_1 + x_1^3 (b_2 x_1^3 + b_3 x_2)] \\ &+ a_3 2 x_2 (b_2 x_1^3 + b_3 x_2) = x^6 (6 b_1 a_1 + b_2 a_2) + \\ &+ x_1^3 x_2 (3 b_1 a_2 + b_3 a_2 + 2 b_2 a_3) + x_2^2 (2 b_3 a_3) \\ &\equiv -x_1^6 - x_2^2 \end{aligned} \tag{51}$$

The following system of three equations with three unknowns is resulted:

$$\begin{bmatrix} 6b_1 & b_2 & 0 \\ 0 & 3b_1 + b_3 & 2b_2 \\ 0 & 0 & 2b_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \tag{52}$$

Applying the Cramer's rule, the unknown a_i coefficients are calculated as:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{12 b_1 b_3 (3b_1 + b_3)} \begin{bmatrix} -6b_1 b_3 - 2b_2^2 - 2b_3^2 \\ 12 b_1 b_2 \\ -18 b_1^2 - 6 b_1 b_3 \end{bmatrix} \tag{53}$$

The Δ for equation (50) is calculated:

$$\begin{aligned} \Delta = a_2^2 - 4a_1 a_3 &= \frac{1}{144 b_1^2 b_3^2 (3b_1 + b_3)^2} \big\{ 144 b_1^2 b_2^2 \\ -48(3b_1 b_3 + b_2^2 + b_3^2)(3b_1^2 + b_1 b_3) \big\} &= \frac{(3b_1 + b_3)^2 + b_2^2}{-3b_1 b_3 (3b_1 + b_3)^2} \end{aligned} \tag{54}$$

Comparing the sign of Δ and a_1 in equations (53) and (54), three cases for sign test of *V(x)* function can be considered as follows:

$$\begin{aligned} &case1: \ \{b_1, b_3 < 0 \ \Rightarrow \ \Delta < 0, \ a_1 > 0 \ \Rightarrow \ v(x) > 0 \\ &case2: \ \{b_1, b_3 > 0 \ \Rightarrow \ \Delta < 0, \ a_1 < 0 \ \Rightarrow \ v(x) < 0 \\ &case3: \ \{b_1, b_3 < 0 \ \Rightarrow \ \Delta > 0 \ \Rightarrow \ v(x) \ is \ sign \ indefinite \end{aligned} \tag{55}$$

Three conditions in equation (55) are in accordance with the three conditions in Theorem 4 and show stability, instability and instability of ZES for equation (49), respectively.

In the following we consider the cases in which the derivatives of basic functions are linearly dependent, meaning that the situation which was studied in Theorem 6 is happened. In this case, the determinant of coefficient matrix in equation (52) is zero. Let the linear combination of derivatives be set identical to zero, then those coefficients are calculated from the following equation:

$$\begin{bmatrix} 6b_1 & b_2 & 0 \\ 0 & 3b_1 + b_3 & 2b_2 \\ 0 & 0 & 2b_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{56}$$

The determinant of equation (56) vanishes when an element of the main diagonal of equation (56) vanishes. In three cases the nonzero a_i coefficients and hence *V(x)* in equation (50) are calculated as following:

$$\mathrm{V}(x) = \begin{cases} x_1^6 = const. \ , & for \quad 6b_1 = 0 \\ b_2 x_1^6 + 2b_3 x_1^3 x_2 = const. , & for \quad 3b_1 + b_3 = 0 \\ (b_2 x_1^3 - 3b_1 x_2)^2 = const. \ , & for \quad 2b_3 = 0 \end{cases} \tag{57}$$

Since the *V(x)* is constant in equation (57), the ZES of equation (49) cannot be asymptotically stable.

Example 7. Find some condition for the following system such that the ZES be asymptotically stable:

$$\begin{cases} \dot{x}_1 = b_{11}x_1^3 + b_{12}x_2 \\ \dot{x}_2 = b_{21}x_1^5 + b_{22}x_1^2 x_2 \end{cases} \tag{58}$$

Solution: Since this system is homogeneous of degree 2 with respect to the weights *r* = (1, 3), the Lyapunov function candidate is selected in the form of equation (50) and its derivative is calculated from the system in equation (58).

$$\begin{aligned} \dot{V}(x) &= a_1 6x_1^5\dot{x}_1 + a_2[3x_1^2 x_2 \dot{x}_1 + x_1^3 \dot{x}_2] + a_3 2x_2\dot{x}_2 \\ &= a_1 6x_1^5(b_{11}x_1^3 + b_{12}x_2) + a_2[3x_1^2 x_2(b_{11}x_1^3 + b_{12}x_2) \\ &+ x_1^3(b_{21}x_1^5 + b_{22}x_1^2 x_2)] + a_3 2x_2(b_{21}x_1^5 + b_{22}x_1^2 x_2) \\ &= (6b_{11}a_1 + b_{21}a_2)x_1^8 + (3b_{12}a_2 + 2b_{22}a_3)x_1^2 x_2^2 + \\ &+ (6b_{12}a_1 + 3b_{11}a_2 + b_{22}a_2 + 2b_{21}a_3)x_1^5 x_2 \end{aligned} \tag{59}$$

It is obvious in equation (59) that $\dot{V}(x)$ cannot be negative definite because with $x_1 = 0$ this function becomes zero. But we can equate it with a negative semi definite function:

$$\begin{aligned} \dot{V}(x) &= (6b_{11}a_1 + b_{21}a_2)x_1^8 + (3b_{12}a_2 + 2b_{22}a_3)x_1^2 x_2^2 + \\ &(6b_{12}a_1 + 3b_{11}a_2 + b_{22}a_2 + 2b_{21}a_3)x_1^5 x_2 \equiv -x_1^8 - x_1^2 x_2^2 \end{aligned} \tag{60}$$

The following system of equations are derived and solved:

$$\begin{bmatrix} 6b_{11} & b_{21} & 0 \\ 0 & 3b_{12} & 2b_{22} \\ 6b_{12} & 3b_{11} + b_{22} & 2b_{21} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{12(b_{12}b_{21} - b_{11}b_{22})(3b_{11} + b_{22})} \times \begin{bmatrix} 2(b_{21}^2 + b_{22}^2) - 6(b_{12}b_{21} - b_{11}b_{22}) \\ -12(b_{12}b_{22} + b_{11}b_{21}) \\ 18(b_{12}^2 + b_{11}^2) - 6(b_{12}b_{21} - b_{11}b_{22}) \end{bmatrix} \tag{61}$$

For the *V(x)* to be positive definite, Δ of equation (50) must be negative:

$$\Delta = a_2^2 - 4a_1 a_3 = \frac{(b_{21} - 3b_{12})^2 + (3b_{11} + b_{22})^2}{3(b_{12}b_{21} - b_{11}b_{22})(3b_{11} + b_{22})^2} < 0 \tag{62}$$

Comparing the equations (61) and (62), the following inequalities are obtained:

$$\begin{cases} b_{12}b_{21} - b_{11}b_{22} < 0 \\ 3b_{11} + b_{22} < 0 \end{cases} \Rightarrow \begin{cases} \Delta < 0 \\ a_1 > 0 \end{cases} \Rightarrow \quad V(x) > 0 \tag{63}$$

Therefore, marginal stability is proved with the Lyapunov theorem for ZES. But with additional condition of $b_{12} = 0$, the LaSalle invariance principle (Slotine and Li, 1991) can be applied and the asymptotic stability can be achieved, because:

$$\begin{aligned} &\dot{V}(x) \equiv 0 \quad \Rightarrow \quad x_1 \equiv 0 \quad \Rightarrow \quad b_{12}x_2 \equiv 0 \\ &if \;\; b_{12} \neq 0 \quad \Rightarrow \quad (x_1, x_2) = (0,0) \end{aligned} \tag{64}$$

CONCLUSION

Lyapunov stability and instability theorems provide strong approaches to the stability analysis of nonlinear systems. This paper combines these theorems based on their common factor which is the negative definiteness of the derivative of the Lyapunov function candidate and introduces the Theorem 4. Hence, this paper focuses on the negative definiteness of the derivative of Lyapunov function instead of positive definiteness of Lyapunov function itself for the purpose of stability analysis. Then, the stability or instability of ZES can be determined by a sign test on the Lyapunov function candidate itself. The root of this approach can also be viewed in the variable gradient method. However the available resources in literature did not introduce variable gradient method as a means to establish instability, and that method was complicated because of the curl condition and the need for integration of ∇V and unavailability of numerical solutions.

In this paper, a method was proposed that has several benefits comparing to other available methods. First, finding certain negative definite function $\dot{V}(x)$ ensures its derivativeness without the complex curl condition. Then, there is no need for integration to calculate of $V(x)$. This method formulates the Lyapunov function candidate as a linear combination of some basic functions. The coefficients of this linear combination should be calculated such that they ensure the negative definiteness of time derivative of that Lyapunov candidate. An algebraic method for construction of this linear combination for zero degree homogeneous systems is provided which can be considered as a generalization of Lyapunov equation for stability analysis of LTI systems.

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