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An Overview of the Theory of Frames and its Application in Signal Transmission

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Abstract: In recent years, the theory of frames that has received a great deal of attention from researchers in the field of functional analysis is being developed and rapidly integrated with other mathematical topics owing to the emergence of its multiple applications in other technical and engineering sciences. In this article, we review the application of this theory in connection with signal transmission after defining the concept of frames and explaining some of the rules and theorems concerning this subject.
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INTRODUCTION

In the study of spaces with scalar multiplication, one of the most important issues is the orthogonal unit basis. This topic causes every element of space to appear as a linear combination of orthogonal basis elements unique. For example, the demonstration of linear and sometimes orthogonal dependence is very difficult or impossible in some cases with regard to scalar multiplication. For the same reason, we need to find a more flexible tool.

Frames are one of these tools. A frame for the scalar multiplication space allows each element of the space to be written as a linear combination of the frame elements while the linear dependence or the orthogonal elements are not specified as conditions. In fact, the frame can be viewed as the generalized bases because it has more elements.

The frames were introduced for the first time in 1952 by Dauphin and Schafer to study some of the issues in the field of non-harmonic Fourier analysis. [5]

Thirty years later in 1986, in their article entitled as “Painless Non-orthogonal Expansions [4], Daubechies, Grossman and Meyer used the frames to find the expansion of function series in L2(R) similar to finding the expansion of functions based on an orthogonal basis.

After the publication of this important article, the subject of frames was widely studied and thus advanced quickly.

The frames are commonly used in the study of signal processing and images to understand the data in a more efficient way. They are also applied in the study of various areas of telecommunications, oil industry, seismology, medicine, physics and biology.

A number of basic rules about frames

Suppose V is a vector space with finite dimensions and its inner multiplication <,> is selected in such a way that it is linear in relation to the first term. We know that the sequence $\{e_k\}_{k=1}^{m}$ in V is a basis for V if the following two conditions are met:

1. V= span$\{e_k\}_{k=1}^{m}$
2. $\{e_k\}_{k=1}^{m}$ is linearly independent, which means if we have $\{c_k\}_{k=1}^{m}$ for numerical coefficients: $\sum_{k=1}^{m} c_k e_k = 0$, then $c_k = 0$ for k=1,2,…,m

As a result of this definition, any f ∈ V has a unique presentation of the basic terms which means the unique numerical coefficients $\{c_k\}_{k=1}^{m}$ exist in such a way that:

(1.1) f =$\sum_{k=1}^{m} c_k e_k$

If $\{c_k\}_{k=1}^{m}$ is an orthogonal unit basis where:

$$< e_k, e_j > = \delta_{k,j} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Then, finding the coefficients of $\{c_k\}_{k=1}^{m}$ will be easy: because of the inner multiplication of F in (1.1) and e_j arbitrary element, we have:

$$< f, e_j > = < \sum_{k=1}^{m} c_k e_k\,, \quad e_j > = \sum_{k=1}^{m} c_k < e_k\,,\, e_j > = \; c_j$$

Therefore:

$$f = \sum_{k=1}^{m} < f, e_k > \; e_k \qquad (1.2)$$

Now, we provide a definition for the frames; below, in the case of 1.1.5, we show that frame $\{f_k\}_{k=1}^{m}$ necessitates a relationship similar to relationship (1.1).

Definition 1.1.1: The countable set of $\{f_k\}_{k\in I}$ from the elements of V is a frame for V, if the constants A, B ≥0 exist so that:

$$A \parallel f \parallel^2 \leq \; \sum_{k\in I} |< f\,, f_k >|^2 \;\leq\; B \parallel f \parallel^2 \quad , \quad \forall f \in V \quad (1.3)$$

Number A and B are called the bounds of the frame. These numbers are not unique. The optimal lower bound is the supremum of all lower bounds of the frame and the optimal upper bound is the infimum of all upper bounds of the frame. Note that the optimal bounds of the frames are also the bounds of the frame. A frame is considered to be normalized if: $\parallel f_k \parallel = 1 \;, \forall k \in I$.

In a vector space with finite dimensions, we can assume that $\{f_k\}_{k\in I}$ family is finite. In this chapter, we will only consider finite families $\{f_k\}_{k=1}^{m} \;\;, \; m \in \mathbb{N}$.

By this limitation and Cauchy - Schwartz inequality, we will have:

$$\sum_{k=1}^{m} |< f\,, f_k >|^2 \;\leq\; \sum_{k=1}^{m} ||f_k||^2\, ||f||^2 \;, \forall f \in V$$

And this condition implies an upper bound for the frame. Regardless, it is possible to find an upper bound that is better than $\sum_{k=1}^{m} ||f_k||^2$. And we will see that the good estimates for the bounds of the frames are significant and important.

To meet the condition of the lower bound in (1.3), span$\{f_k\}_{k=1}^{m} = V$ is necessary. This state is sufficient to establish (1.3); in fact, any finite sequence is a frame for its production space.

Theorem 1.1.2: We suppose $\{f_k\}_{k=1}^{m}$ is a sequence in V. Then, $\{f_k\}_{k=1}^{m}$ is a frame for span$\{f_k\}_{k=1}^{m}$.

Proof: It can be assumed that all f_k are not zero. As observed, the condition of the upper bound of the frame is established by B=$\sum_{k=1}^{m} ||f_k||^2$. Now, we have: W= span$\{f_k\}_{k=1}^{m}$ and we consider the below continuous case:

$$\phi: W \;\longrightarrow\; \mathbb{R} \quad , \qquad \phi(f) := \sum_{k=1}^{m} |< f, f_k >|^2$$

Orb unit is compact in W, therefore, we may find g∈w with || g || = 1 in such a way that:

$$A = \sum_{k=1}^{m} |< g\,, f_k >|^2 = \inf \{ \sum_{k=1}^{m} |< f\,, f_k >|^2 \;:\; f \in w \,, \parallel f \parallel = 1 \}$$

Obviously, A> 0. Now, given f∈W and f ≠ 0, we have:

$$\sum_{k=1}^{m} |< f\,, f_k >|^2 = \sum_{k=1}^{m} |< \frac{f}{||f||}, f_k >|^2 \, ||f||^2 \geq A||f||^2$$

Result 1.1.3: $\{f_k\}_{k=1}^{m}$ family from the elements of V is a frame for V if and only if: V= span$\{f_k\}_{k=1}^{m}$

1.1.3 Result shows that the number of elements in a frame might be more than the number needed for it to be a basis. In particular, if $\{f_k\}_{k=1}^{m}$ is a frame for V and $\{g_k\}_{k=1}^{n}$ is a finite arbitrary set from V vectors, then: $\{f_k\}_{k=1}^{m} \cup \{g_k\}_{k=1}^{n}$ will be a frame for V. A frame which is not a basis in called complete supremum.

Now, we consider the V vector space with $\{f_k\}_{k=1}^{m}$ and we define the below relation:

$$T : \mathbb{C}^m \;\longrightarrow V \,, T\{c_k\}_{k=1}^{m} = \; \sum_{k=1}^{m} c_k f_k \qquad (1.4)$$

T is usually called a pre-frame operator or the combination operator. Its adjoint operator is defined as follows:

$$T^*: V \rightarrow \; \mathbb{C}^m \,, T^* f = \{< f, f_k >\}_{k=1}^{m} \qquad (1.5)$$

And it is called the analysis operator. We will have S frame by combining T and its adjoint operator T*:

$$S\!: V \rightarrow V \,, Sf = T\,T^* f = \; \sum_{k=1}^{m} < f, f_k > f_k \qquad (1.6)$$

Note that we would have the following from the terms of frame operator:

$$< Sf, f > = \sum_{k=1}^{m} |< f, f_k >|^2 \;\; , \qquad f \in V\,; \qquad (1.7)$$

The condition of the lower bound of the frame can be somehow considered as the lower bound of the frame operator.

We refer to $\{f_k\}_{k=1}^m$ as narrow if we set A=B in the definition of the frame:

$$\sum_{k=1}^{m} |< f, f_k >|^2 = A||f||^2 , \qquad f \in V ; \qquad (1.8)$$

For a narrow frame, the amount of A in (1.8) is called the bound frame.

Theorem 1.1.4: If $\{f_k\}_{k=1}^m$ is a narrow frame with bound A for V, then S = AI (where I is the identity operator in V) and:

$$f = \frac{1}{A}\sum_{k=1}^{m} < f, f_k > f_k \quad , \forall f \in V \qquad (1.9)$$

One explanation for (1.9) relationship is that if $\{f_k\}_{k=1}^m$ is a narrow frame and we want f ∈V to be expressed as this linear combination $\sum_{k=1}^m c_k f_k$, we can simply define g_k as $g_k = \frac{1}{A} f_k$ and we have: $c_k = < f, g_k >$. Equation (1.9) is similar to (1.2) display by an orthogonal unit basis with the difference that the relationship (1.9) has $\frac{1}{A}$ coefficient. Now, as for conventional frames, we show that for any f ∈V by adopting the proper sequence $\{g_k\}_{k=1}^m$ we would have a display in the form of f $=\sum_{k=1}^m < f, g_k > f_k$. The theorem is derived from one of the most important results of the frames in which the relationship (1.10) is called the frame decomposition.

Theorem 1.1.5: We suppose that $\{f_k\}_{k=1}^m$ is a frame for V with S frame operator. Then:

(i) S is invertible and self-adjoint

(ii) Every f ∈V can be represented as follows:

f $=\sum_{k=1}^m < f, S^{-1} f_k > f_k = \sum_{k=1}^m < f, f_k > S^{-1} f_k \qquad (1.10)$

(iii) If f ∈V also has f= $\sum_{k=1}^m c_k f_k$ for numerical coefficients $\{c_k\}_{k=1}^m$, then:

$$\sum_{k=1}^m |c_k|^2 = \sum_{k=1}^m |< f, S^{-1} f_k >|^2 + \sum_{k=1}^m |c_k - < f, S^{-1} f_k >|^2$$

Proof: Since S = TT*, it is clear that S is self-adjoint. Now, we prove that S is one-to-one. We suppose that F ∈V and we assume that: Sf = 0, then:

0=$< Sf, f >= \sum_{k=1}^m |< f, f_k >|^2$

And so it follows that f = 0. The one-to-one quality of S implies its spanning, but it is better to provide a direct proof. From the frame feature as a result of 1.1.3 we have span$\{f_k\}_{k=1}^m = V$. With an obvious f ∈V, we can find g∈V so that Tg = f; with the selection of $g \in \mathcal{N}_T^{\perp} = \mathcal{R}_{T^*}$

It follows that $\mathcal{R}_S = \mathcal{R}_{TT^*} = V$. Therefore, S is surjective. Any F ∈V is shown in the following way:

$$f = SS^{-1}f$$
$$= TT^* S^{-1} f$$
$$= \sum_{k=1}^m < S^{-1} f, f_k > f_k$$

Since S is self-adjoint, we have:

$$f = \sum_{k=1}^{m} < f, S^{-1} f_k > f_k$$

The second presentation in (1.10) is obtained as seen below by means of $f = S^{-1}Sf$

To prove the third part, we assume f= $\sum_{k=1}^m c_k f_k$, and we can write:

$$\{c_k\}_{k=1}^m = \{c_k\}_{k=1}^m - \{< f, S^{-1} f_k >\}_{k=1}^m + \{< f, S^{-1} f_k >\}_{k=1}^m$$

And we will have: $\sum_{k=1}^m (c_k - < f, S^{-1} f_k >) f_k = 0$ which means:

$\{c_k\}_{k=1}^m - \{< f, S^{-1} f_k >\}_{k=1}^m \in \mathcal{N}_T = \mathcal{R}_T^{\perp}$ and because:

$\{< f, S^{-1} f_k >\}_{k=1}^m = \{< S^{-1} f, f_k >\}_{k=1}^m \in \mathcal{R}_{T^*}$, therefore, (iii) is established.

Each frame in a space with finite dimensions has a subfamily which forms a basis. If $\{f_k\}_{k=1}^m$ is a frame, but is not a basis, then it is below the non-zero sequence $\{d_k\}_{k=1}^m$ in a way that: $\sum_{k=1}^m d_k f_k = 0$ so f ∈V can be written as below:

$$f = \sum_{k=1}^{m} < f, S^{-1} f_k > f_k + \sum_{k=1}^{m} d_k f_k$$

$$\sum_{k=1}^m (< f, S^{-1} f_k > + d_k) f_k \qquad =$$

Therefore, f frame can be displayed in multiple forms with combinations of the frame elements Theorem 1.1.5 shows that the coefficients of $\{< f, S^{-1} f_k >\}_{k=1}^m$ have the minimum amount with ℓ^2 norm among all sequences of $\{c_k\}_{k=1}^m$ in such a way that: f= $\sum_{k=1}^m c_k f_k$

Numbers $< f, S^{-1}f_k >$, $k = 1, \dots, m$ are called the coefficients of the frame. Note that because S: V → V is bijective, as a result of 1.1.3 the sequence $\{S^{-1}f_k\}_{k=1}^{m}$ is also a frame which is called dual focus $\{f_k\}_{k=1}^{m}$. The case of 1.1.5 provides us with specific information in the state of being a basis $\{f_k\}_{k=1}^{m}$.
Result 1.1.6: Suppose $\{f_k\}_{k=1}^{m}$ is a basis for V. Then, the unique family of $\{g_k\}_{k=1}^{m}$ is in V so that:
(1.11) $f = \sum_{k=1}^{m} < f, g_k > f_k \quad , \forall f \in V$

In the language of the frame operator, we have: $\{g_k\}_{k=1}^{m} = \{S^{-1}f_k\}_{k=1}^{m}$. In addition, $< f_j, g_k > = \delta_{j,k}$.
Proof: The existence of family $\{g_k\}_{k=1}^{m}$ in a way that is established in (1.11) is concluded from the case (1.1.5). The proof of uniqueness is left to the reader. By applying relation (1.11) in the constant member f_j and by taking into account that $\{f_k\}_{k=1}^{m}$, the following relation is derived: $< f_j, g_k > = \delta_{j,k}$ for k=1,2,…,m □

Intuitively, it can be shown why frames are important in the transmission of frequency. For further study, one may refer to [6]. Suppose we want to transfer f signal which belongs to V vector space from transmitter A to receiver R. If both A and R have the data from frame $\{f_k\}_{k=1}^{m}$for V, then this can be done by transferring the coefficients of the frame, which are $\{< f, S^{-1}f_k >\}_{k=1}^{m}$ by A. Based on the information from these numbers, receiver R can remake f signal through the analysis of the frame. Now, suppose R receives a disturbed signal which is a disorder in$\{< f, S^{-1}f_k > +c_k\}_{k=1}^{m}$, and the exact coefficients of the frame. Based on the received coefficients, R shows that the transmitted signal is as follows:
$\sum_{k=1}^{m}(< f, S^{-1}f_k > +c_k)f_k = \sum_{k=1}^{m} < f, S^{-1}f_k > f_k + \sum_{k=1}^{m} c_k f_k = f + \sum_{k=1}^{m} c_k f_k$

The contrast with the accurate f signal is in $\sum_{k=1}^{m} c_k f_k$. If $\{f_k\}_{k=1}^{m}$ and is complete, then the pre-frame operator $T\{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k f_k$ has an unclear core and the existence of the disorder section may be aggregated with zero and become zero. This case would never occur if, $\{f_k\}_{k=1}^{m}$ is an orthogonal unit basis. In this state, $||\sum_{k=1}^{m} c_k f_k||^2 = \sum_{k=1}^{m} |c_k|^2$, therefore the disorder section will exacerbate the modernization.
Earlier, we had seen that for f ∈V, $\{< f, S^{-1}f_k >\}_{k=1}^{m}$ coefficients of the frames have the lowest value among all sequences with ℓ^2 norm between all the sequences of $\{c_k\}_{k=1}^{m}$ for which $f = \sum_{k=1}^{m} c_k f_k$. Sometimes, we want this value to be the lowest value with regard to norms other than ℓ^2. Now, the coefficients are shown with the least values of ℓ^2 norm.
Theorem 1.1.7: Suppose $\{f_k\}_{k=1}^{m}$ is a frame for V vector space with finite dimensions and f ∈V, then, the coefficients $\{d_k\}_{k=1}^{m} \in \mathbb{C}^m$ are present in such a way that: $f = \sum_{k=1}^{m} d_k f_k$ and

$$\sum_{k=1}^{m} |d_k| = \inf\left\{\sum_{k=1}^{m} |c_k| : f = \sum_{k=1}^{m} c_k f_k\right\} \qquad (1.12)$$

Proof: we assume that f ∈V is constant. Obviously, the coefficients of $\{c_k\}_{k=1}^{m}$ can be provided in such a way that: $f = \sum_{k=1}^{m} c_k f_k$

We establish that r = $\sum_{k=1}^{m} |c_k|$, as we want to minimize the coefficients with ℓ^1 norm. Then, it is also clear that we can limit our search for a minimizer to sequence $\{d_k\}_{k=1}^{m}$ which belongs to the below compact set:
$M = \{\{d_k\}_{k=1}^{m} \in \mathbb{C}^m \ : \ |d_k| \le r , \ k = 1, \dots, m$
Now, the result is derived from the fact that the set $\{\{d_k\}_{k=1}^{m} \in M \mid f = \sum_{k=1}^{m} d_k f_k\}$ is compact and function $\phi : \mathbb{C}^m \to \mathbb{R} , \phi\{d_k\}_{k=1}^{m} = \sum_{k=1}^{m} |d_k|$ is continuous.

There are several important differences between 1.1.5 and 1.1.7 cases. In the case of 1.1.5, we clearly found the sequence to minimize the coefficients with ℓ^2 norm in f expansion which was unique and linearly dependent on f. On the other hand, the case of 1.1.7 only indicates the existence of a minimizer with ℓ^1 norm that may not be unique. Even if it is unique, it might not be the linearly dependent on f. An algorithm to find a minimizer with ℓ^1 norm can be found in [7].

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